

A NUMERICAL SOLUTION OF LAMINAR FLOW IN PORUS MEDIA WITH TRIANGULAR DUCT BY FINITE DIFFERENCE METHOD WITH MATLAB

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ABSTRACT

The fully developed laminar flow of a various in-compressible liquid in along triangular duct has Figure been investigated. The generalized momentum equation which governs the flow has been solved by finite difference method using MATLAB. The numerical results are given.

KEYWORDS: Matlab, Generalized Momentum Equation, Triangular Duct

INTRODUCTION

It is well known that fluid flow through porous media plays important role in hydrology, petroleum engineering, chemical engineering, bio-chemical engineering, agricultural engineering, medicines and paper technology etc. Henry Darcy [2] initiated the mathematical theory of the flow of a fluid through porous medium. The generalized Darcy's law states that the seepage velocity of the fluid is proportional to the pressure gradient. For slowly spatially varying flow, Darcy's is surely correct. If, however, the flow involves large shear, one expect further terms involving velocity gradient to appear. Brinkman [1] put Darcy's law in a better theoretical shape by taking in to account the effect of viscous stress. This generalised Darcy's law gave results in the case of highly porous media.

The problem of fully developed flow through square and rectangular shaped ducts filled with porous media or in absence of it has been studied by many scientist - Ramchandra and Spalding [5], Wong [9], Patil [4], Sharma [7] has solved the problem for rectangular duct by using finite Fourier sine transform.

STATEMENT OF THE PROBLEM

To obtain the solution of the problem of laminar flow in porous media with porous triangular duct, consider here a rectangular duct with its mathematical formulation. This mathematical formulation is in terms of velocity component (w) with appropriate boundary conditions for rectangular element. The governing equation of the problem is nonlinear partial differential equation. Using finite difference method, the solution has been obtained by taking triangular element (ΔBCD) as shown in the following Figure [1].

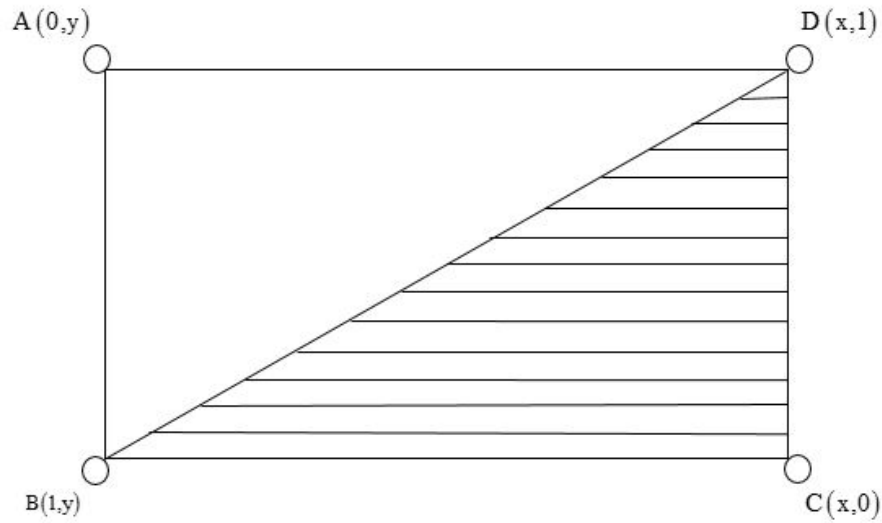


Figure 1

FORMULATION OF THE PROBLEM

Consider a rectangular duct of width 'a' and length 'b' filled with porous material of permeability K . A Cartesian co-ordinate system is used in such a way that the wall of the rectangular is at $x = 0, x = a, y = 0, y = b$ and the flow of the incompressible viscous fluid inside the duct is assumed to be fully developed. All the four boundaries are impermeable to fluid flow. Permeability K of the porous medium and all fluid properties are taken to be constant. The velocity components in the directions of X, Y and Z are assumed to be the form

$$W = 0, V = 0 \text{ and } U = U(x, y) \text{ Respectively.} \quad (1)$$

The equation of continuity

$$\nabla \vec{V} = 0 \quad (2)$$

Is satisfied identically by the form of velocity component (1). The equation of the motion of a viscous fluid through porous medium as proposed by brinkman [] is

$$\nabla P = \mu \vec{\nabla}^2 V - \vec{V} \quad (3)$$

Where \vec{V} is the velocity vector, μ is the viscosity of the fluid and K is the permeability constant of the medium.

The equation of motion (3) for the velocity component of the form (1), becomes

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{U}{K} = \frac{1}{\mu} \frac{dp}{dz} \quad (4)$$

Where $\frac{dp}{dz}$ is the constant pressure gradient?

The boundary condition of the problem is that the velocity component U is zero at all the four face of the

rectangular duct. Hence

$$U(0, y) = U(a, y) = U(x, 0) = U(x, b) = 0 \tag{5}$$

Introducing the dimensionless variables defined by

$$x = ax, y = by, U = U_1 u \tag{6}$$

$$\text{Where, } U_1 = a^2 \left(-\frac{1}{\mu} \right) \frac{dp}{dz} \tag{7}$$

The transformed equation with boundary condition as

$$\frac{\partial^2 U}{\partial x^2} + \alpha^2 \frac{\partial^2 U}{\partial y^2} - \delta^2 U + 1 = 0 \tag{8}$$

And

$$U(0, y) = U(1, y) = U(x, 0) = U(x, 1) = 0 \tag{9}$$

Where, $\alpha = \frac{a}{b}$ is the aspect ratio and $\delta = \frac{a}{\sqrt{k}}$ is the permeability parameter.

SOLUTION BY FINITE DIFFERENCE METHOD

$$\frac{\partial^2 U}{\partial x^2} + \alpha^2 \frac{\partial^2 U}{\partial y^2} = \delta^2 U - 1 \tag{10}$$

Consider finite difference standard five point formula

$$\text{Let } U(x, y) = U_{i,j} \tag{11}$$

$$U_{xx} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} + o(h^2) \tag{12}$$

$$U_{yy} = \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{k^2} + o(k^2) \tag{13}$$

Consider a rectangular region R for which $U(x, y)$ is known at the boundary. Divide this region in to a network of square mesh of side h, as shown in Figure [2] (assuming that an exact subdivision of R is possible). Replacing those derivatives in equation number (10) by their difference approximations, we have

$$\frac{1}{h^2} (U_{i-1,j} - 2U_{i,j} + U_{i+1,j}) + \frac{\alpha^2}{k^2} (U_{i,j-1} - 2U_{i,j} + U_{i,j+1}) = \delta^2 U_{i,j} - 1 \tag{14}$$

This shows that the value of U at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. Equation number (14) is called the standard five point formula which is exhibited in Figure

[2] given below.

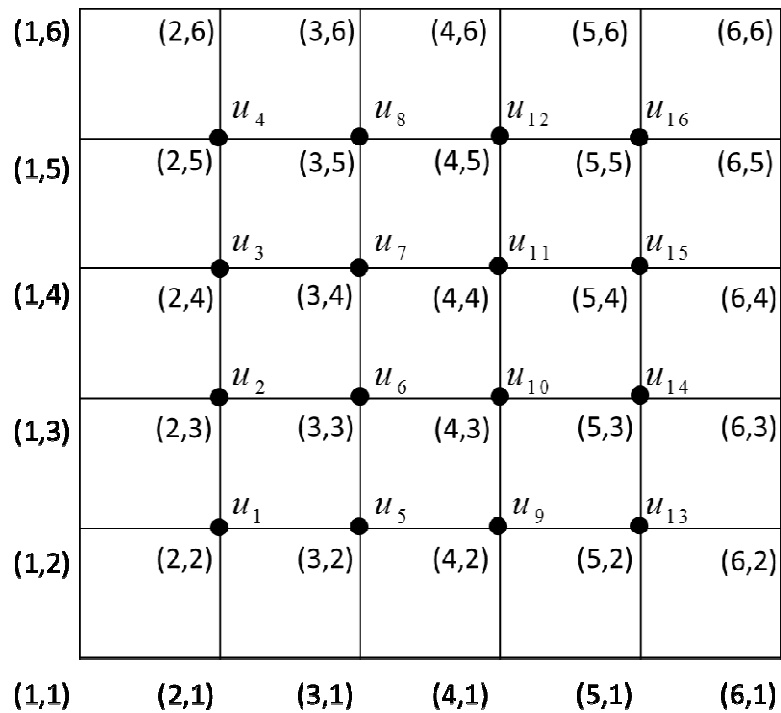


Figure 2

Now take

$$n = 5, h = \frac{b-a}{n} = k$$

$$h = 0.2, k = 0.2$$

$$u_1 = U_{2,2}$$

$$u_5 = U_{3,2}$$

$$u_9 = U_{4,2}$$

$$u_{13} = U_{5,2}$$

$$u_2 = U_{2,3}$$

$$u_6 = U_{3,3}$$

$$u_{10} = U_{4,3}$$

$$u_{14} = U_{5,3}$$

$$u_3 = U_{2,4}$$

$$u_7 = U_{3,4}$$

$$u_{11} = U_{4,4}$$

$$u_{15} = U_{5,4}$$

$$u_4 = U_{2,5}$$

$$u_8 = U_{3,5}$$

$$u_{12} = U_{4,5}$$

$$u_{16} = U_{5,5}$$

From the Equation number (14), we can write the equations for all (i, j) , where $i, j = 2, 3, 4, 5$ by taking $\alpha = 0.5$ & $\delta = 1.0$, which is given by

$$2.54u_1 - 0.25u_2 - u_5 = 0.04 \quad (15)$$

$$-0.25u_1 + 2.54u_2 - 0.25u_3 - u_6 = 0.04 \quad (16)$$

$$-0.25u_2 + 2.54u_3 - 0.25u_4 - u_7 = 0.04 \quad (17)$$

$$-0.25u_3 + 2.54u_4 - u_8 = 0.04 \quad (18)$$

$$-u_2 - 0.25u_5 + 2.54u_6 - 0.25u_7 - u_{10} = 0.04 \tag{19}$$

$$-u_4 - 0.25u_7 + 2.54u_8 - u_{12} = 0.04 \tag{20}$$

$$-u_5 + 2.54u_9 - 0.25u_{10} - u_{13} = 0.04 \tag{21}$$

$$-u_6 - 0.25u_9 + 2.54u_{10} - 0.25u_{11} - u_{14} = 0.04 \tag{22}$$

$$-u_7 - 0.25u_{10} + 2.54u_{11} - 0.25u_{12} - u_{15} = 0.04 \tag{23}$$

$$-u_8 + 2.54u_{11} - 0.25u_{12} - u_{16} = 0.04 \tag{24}$$

$$-u_9 + 2.54u_{13} - 0.25u_{14} = 0.04 \tag{25}$$

$$-u_{10} - 0.25u_{13} + 2.54u_{14} - 0.25u_{15} = 0.04 \tag{26}$$

$$-u_{11} - 0.25u_{14} + 2.54u_{15} - 0.25u_{16} = 0.04 \tag{27}$$

$$-u_{12} - 0.25u_{15} + 2.54u_{16} = 0.04 \tag{28}$$

Equation number (15) to (28) gives the system which is represented by

$$AU = B$$

Where,

$$A = \begin{pmatrix} 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.25 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.25 & 2.54 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -0.25 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & -0.25 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & -0.25 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2.54 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & -0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 & -0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.25 & 2.54 \end{pmatrix}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \\ 0.04 \end{pmatrix}$$

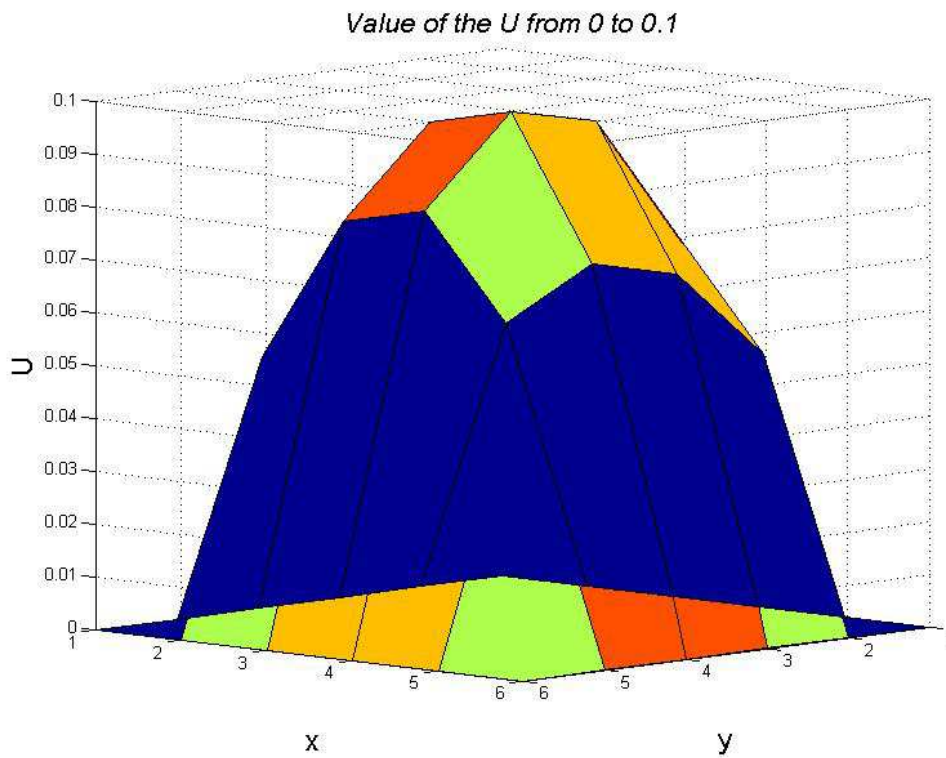
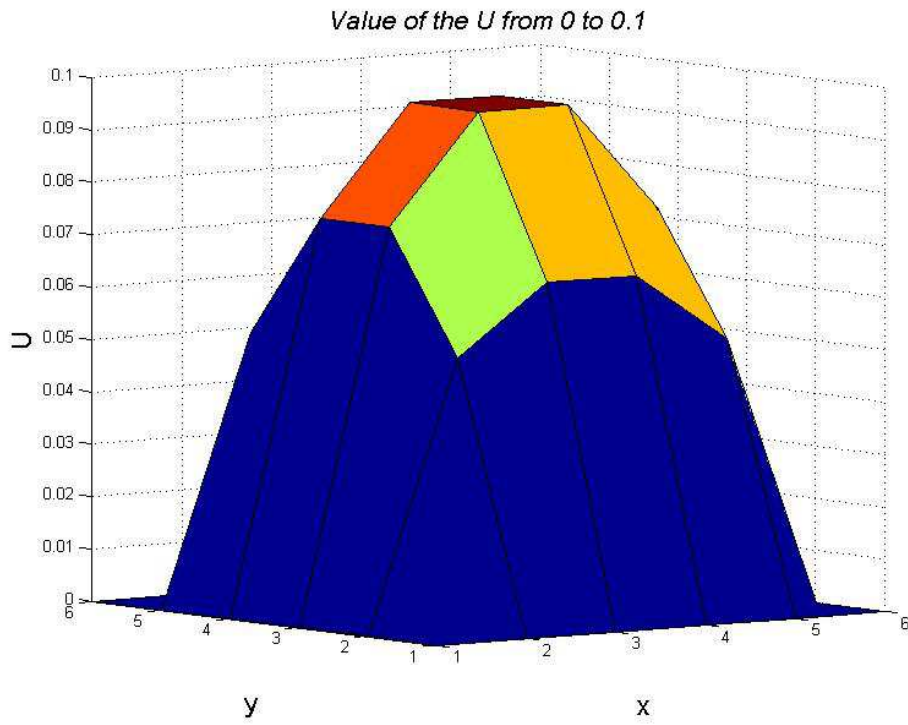
Solve the given system by simply the inverse method

i.e. $U = A^{-1}B$

We get the values for the nodes u_1 to u_{16}

| | | | |
|----------------|----------------|-------------------|-------------------|
| $u_1 = 0.0517$ | $u_5 = 0.0751$ | $u_9 = 0.0751$ | $u_{13} = 0.0517$ |
| $u_2 = 0.0650$ | $u_6 = 0.0960$ | $u_{10} = 0.0960$ | $u_{14} = 0.0650$ |
| $u_3 = 0.0650$ | $u_7 = 0.0960$ | $u_{11} = 0.0960$ | $u_{15} = 0.0650$ |
| $u_4 = 0.0517$ | $u_8 = 0.0751$ | $u_{12} = 0.0751$ | $u_{16} = 0.0517$ |

Figure [3] to Figure [6] shows the 3D surface for solution of the problem from different sides.



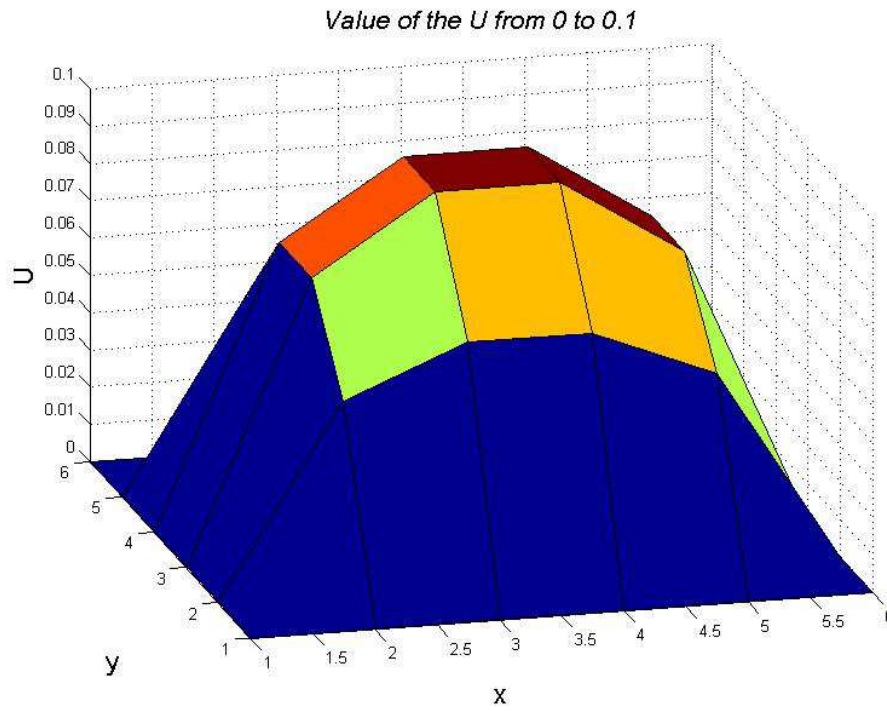


Figure 5

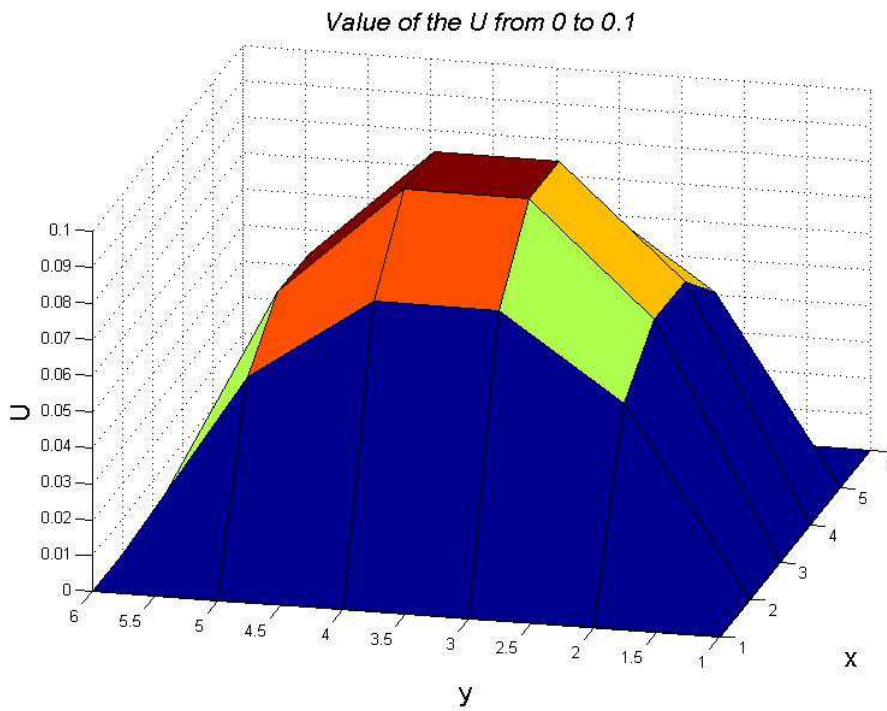


Figure 6

CONCLUSIONS

Value of velocity component U at x and y at different nodes or points are given in the following table.

Table [1], we fix “X” and find the value of “U” for the different value of “Y”. Whose graph is defined in the Figure [7].

Table 1

| Value of U | Value at X | | | | | |
|------------|------------|--------|--------|--------|--------|---|
| | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.2 | 0 | 0.0517 | 0.0751 | 0.0751 | 0.0517 | 0 |
| 0.4 | 0 | 0.065 | 0.096 | 0.096 | 0.065 | 0 |
| 0.6 | 0 | 0.065 | 0.096 | 0.096 | 0.065 | 0 |
| 0.8 | 0 | 0.0517 | 0.0751 | 0.0751 | 0.0517 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

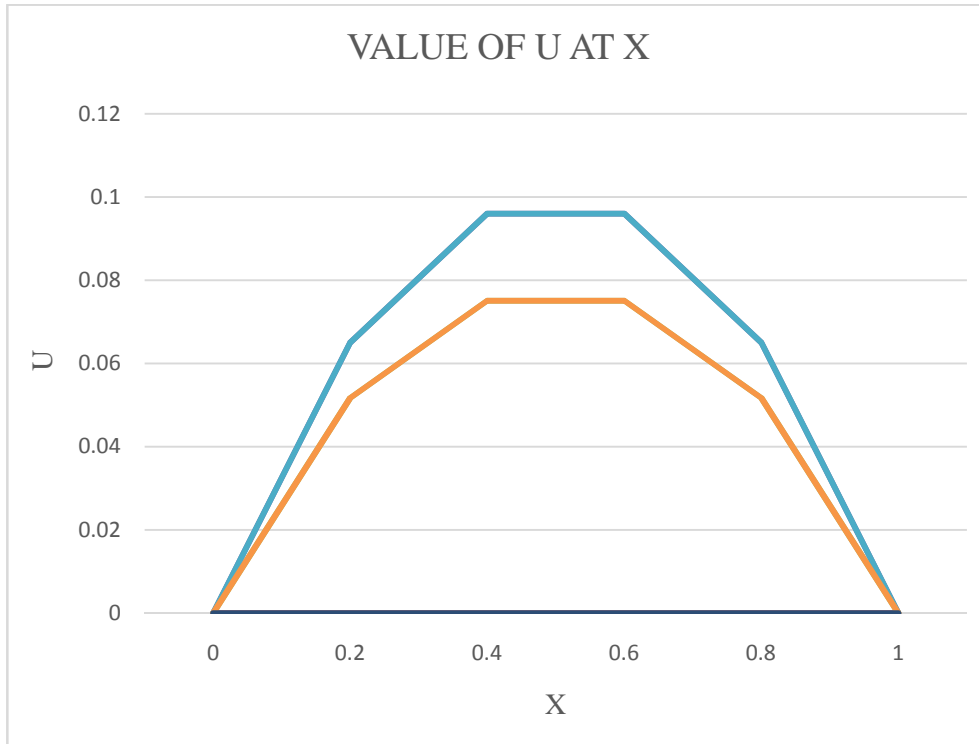


Figure 7

Table [2], we fix “Y” and find the value of “U” for the different value of “X”. Whose graph is defined in the Figure [8].

Table 2

| Value of U | Value at Y | | | | | |
|------------|------------|--------|-------|-------|--------|---|
| | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.2 | 0 | 0.0517 | 0.065 | 0.065 | 0.0517 | 0 |
| 0.4 | 0 | 0.0751 | 0.096 | 0.096 | 0.0751 | 0 |
| 0.6 | 0 | 0.0751 | 0.096 | 0.096 | 0.0751 | 0 |
| 0.8 | 0 | 0.0517 | 0.065 | 0.065 | 0.0517 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

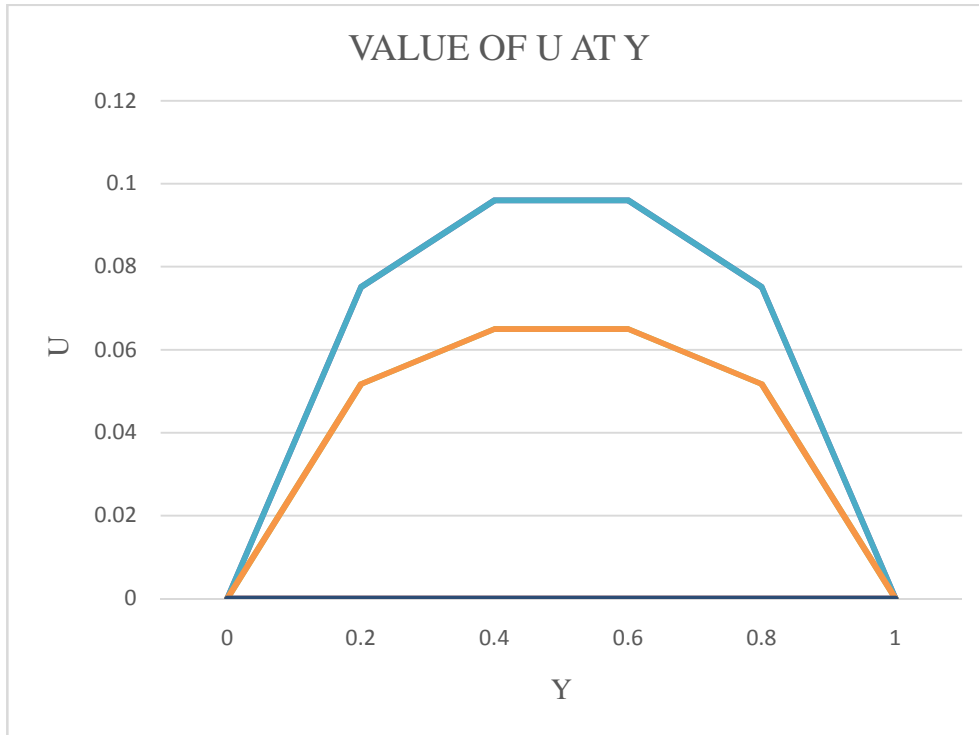


Figure 8

Here we can observe that the value of U at different points of (X, Y) first increased, reached at maximum value and then decreased. Whose graph becomes concave downward, this can be shown in the above figures, Figure [7] and Figure [8]

Now we can more generalize the problem by making more iteration and the solution is clear. For that I used MATLAB programming and get the nearest solution of the problem.

MATLAB PROGRAMME OF THE PROBLEM

Here I define the general MATLAB programming for the given problem in which by guessing very small step size (the value of m and n), we get very small value of h and k . that means increasing the meshes gives the large number of nodes (grid points). Also we can do the same for the different boundary and the result will be clear for us.

The MATLAB programme of the problem is given by

```
Clc
```

```
Clear all
```

```
a=0;
```

```
b=1;
```

```
h=0.01;
```

```
n=(b-a)/h;
```

```
c=0;
```

```

d=1;

k=0.01;

m=(d-c)/k;

al=0.5;

dl=1.0;

u(1,1)=0;

for l=2:1:n+1

u(l,1)=0;

u(l,m+1)=0;

end

for p=2:1:m+1

u(1,p)=0;

u(n+1,p)=0;

end

for q=1:1:1000

for i=2:1:n

for j=2:1:m

u(i,j)=(1./((2./h.^2)+2.*(al.^2./k.^2)+dl.^2)).*((1./h.^2).*(u(i-1,j)+u(i+1,j)))+(al.^2./k.^2).*(u(i,j-1)+u(i,j+1))+1);

end

end

end

w=u;

surf(w)

```

This is the programme for the given problem in which (a, b) and (c, d) denotes the boundary points for the mesh, n and m is the number of mesh you taken which gives the value of h and k means length between the grid points respectively. And boundary value also defined. Also here we assumed the value of $al = \alpha$ and $dl = \delta$. q Defines the number of iteration, by taking large number of iteration we get the nearer solution of the problem. W gives the final value of the nodes u_i .

Now I take the large value of $m=100$ and $n=100$ gives the small value of $h=0.01$ and $k=0.01$ respectively. This gives the total 9801 nodes means u_1 to u_{9801} . Whose values are find using MATLAB. For that the solution is defined in

the following figure. Figure [9] shows the 2D- plot for the solution of the problem. And Figure [10] to Figure [13] shows the 3D surface for the solution of the problem.

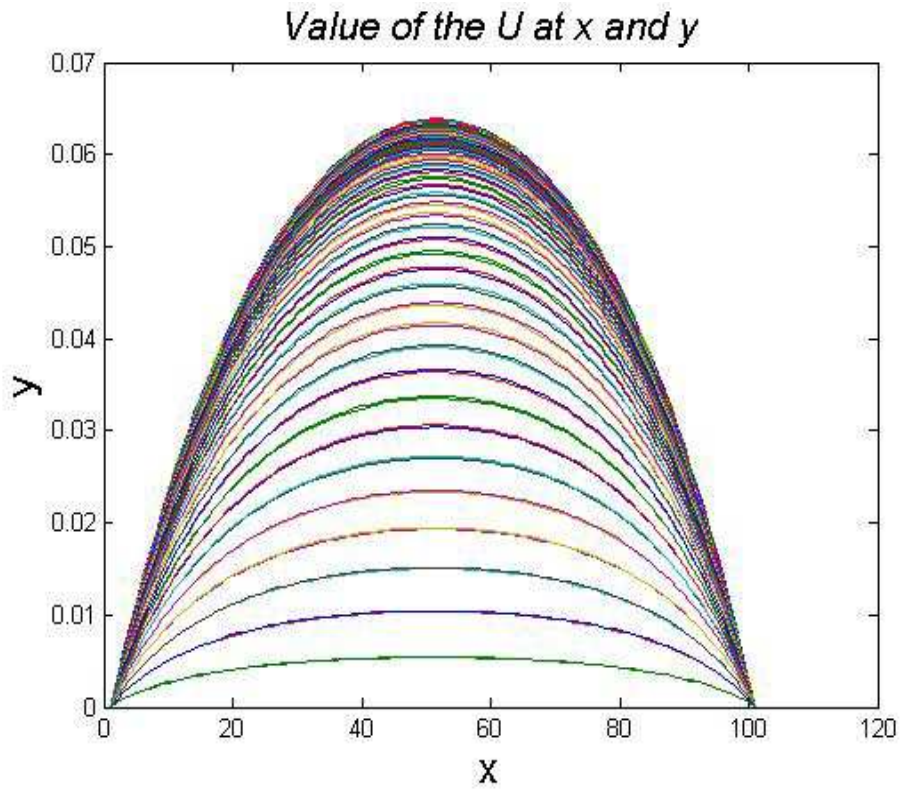


Figure 9

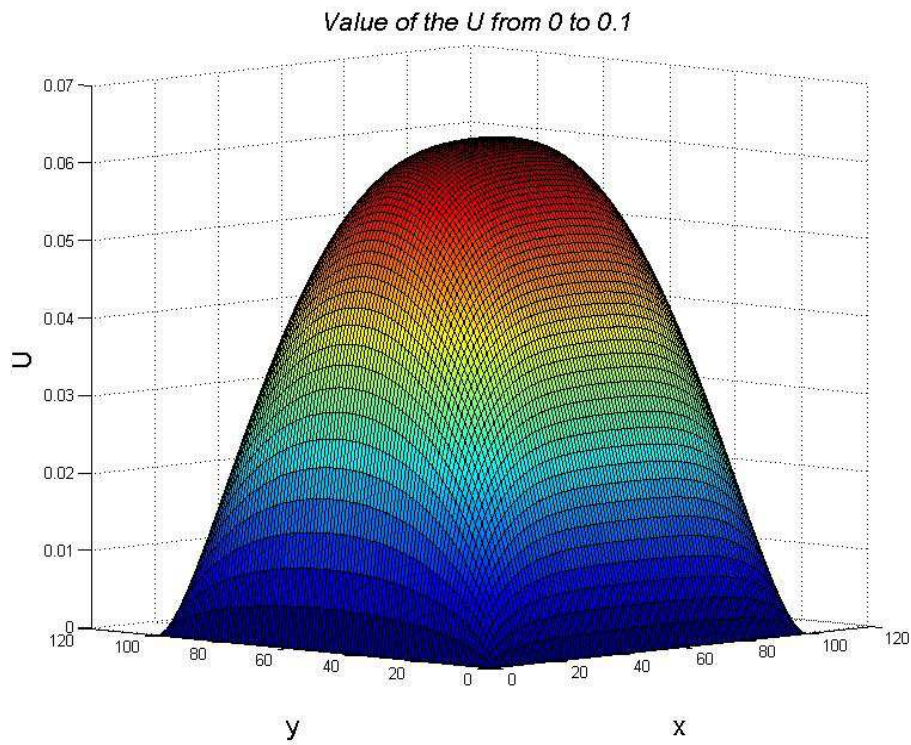


Figure 10

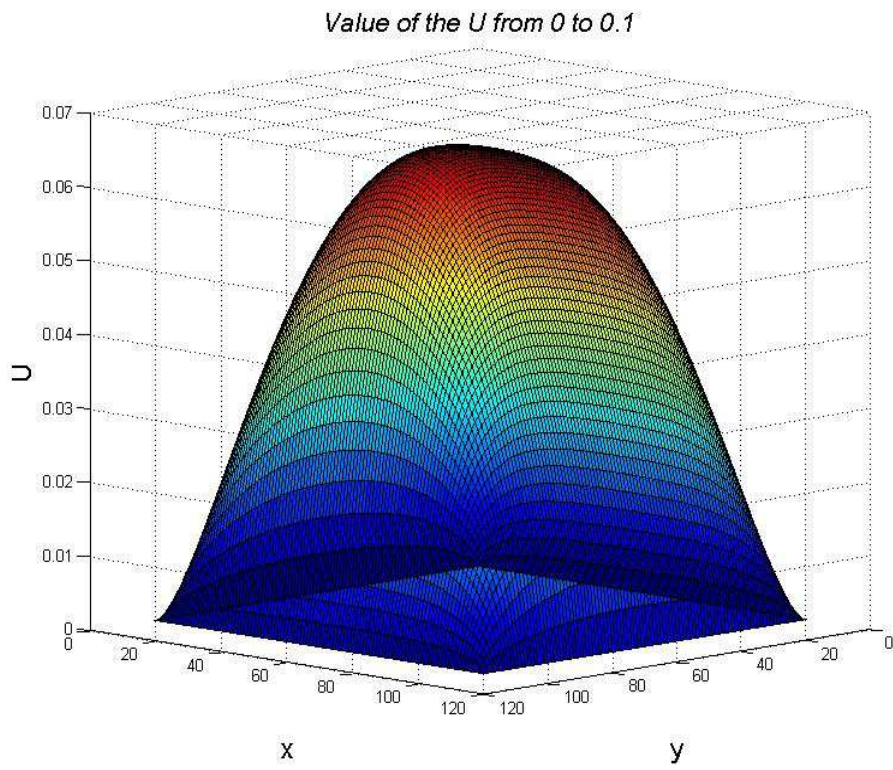


Figure 11

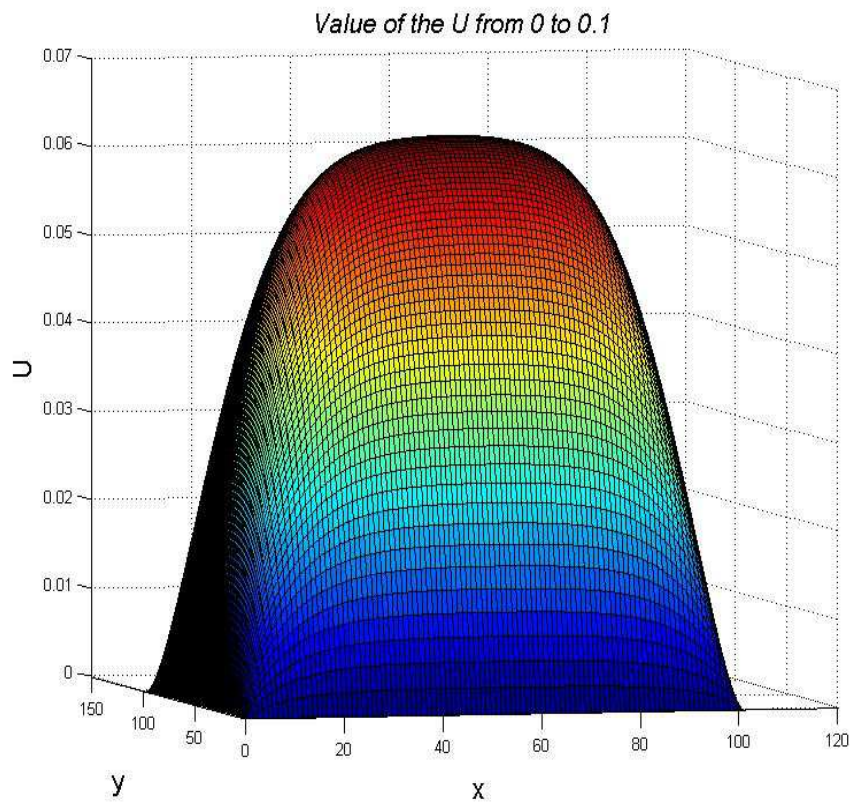


Figure 12

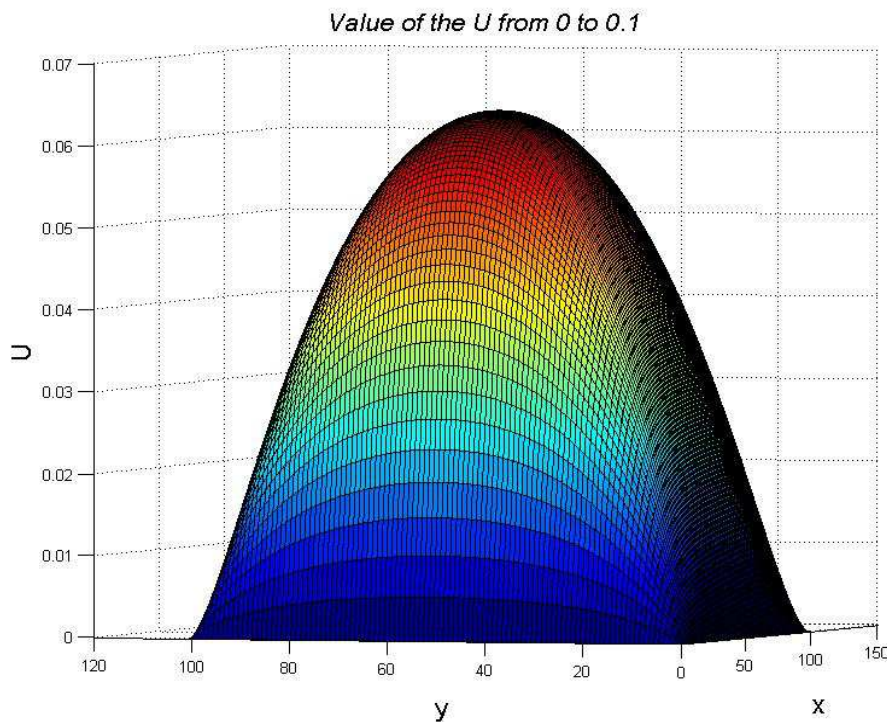


Figure 13

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